ON THE COMMUTATOR GROUPS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society at its Fourth Summer Meeting, Toronto, Canada, August 17, 1897.)

THE operator $s^{-1}t^{-1}st$ has been called by Dedekind the commutator of s and t. When s and t are commutative this commutator is identity and *vice versa*. Since $s^{-1}t^{-1}st = (t^{-1}s^{-1}ts)^{-1}$, the inverse of a commutator is a commutator of the same two operators and may be obtained by merely interchanging the operators of the given commutator. Hence s and t have, in general, two commutators, the one being the inverse of the other. These two commutators are identical only when their order is 2 or 0.

When s and t represent successively all the operators of a group (G) their commutators generate a subgroup (G_1) of G. Since G_1 contains all the conjugates of any one of these generators multiplied into its inverse, these products being commutators of G, it must also contain all the given conjugates. Hence it is a self-conjugate (invariant) subgroup of G.

The quotient group of G with respect to G_1 is isomorphic to G and of order $g \div g_1$, g and g_1 being the orders of G and G_1 respectively. Since the commutators of the corresponding operators in these isomorphic groups must correspond, all those of the quotient group must be identity; that is, this quotient group is an Abelian or commutative group. If G has any other Abelian quotient group its self-conjugate subgroup which corresponds to identity in this quotient group must include G_1 since the commutator subgroup of an Abelian group is identity.

G_1 is a characteristic* subgroup of G, for the latter has only one commutator subgroup. If we represent the commutator subgroup of G_1 by G_2, that of G_2 by G_3, etc., it follows that each of the following groups

$$G, G_1, G_2, G_3, \cdots$$

is not only a characteristic subgroup of the one which immediately precedes it, but also of all those which precede it. Following Lie's notation, we may call G_1, G_2, G_3, $\cdots$ the first, second, third, $\cdots$ derived of G; the first derived and the commutator subgroup being equivalent terms.

*Cf. Frobenius, *Sitzungsberichte der Berliner Akademie*, 1895, p. 183.

Let G_a represent the first of these groups which satisfies the equation

$$G_a = G_{a+1}$$

It must then be identical with all those which follow it in the given series. We may therefore suppose that it terminates this series. If it is not identity it cannot be solvable, for its factors of composition must include composite numbers. In this case G is not solvable.* When $G_a = 1$, G is evidently solvable.† Hence *the necessary and sufficient condition that a group is solvable is that we arrive at unity by forming its successive derived subgroups.*‡

When a group, like G_a, is identical with its commutator subgroup it has been called by Lie a perfect group. The necessary and sufficient condition that a group is perfect is that is not isomorphic to any Abelian group. Hence every simple group of a composite order must be perfect. The composite group formed by the product of any number of such simple groups is clearly also perfect.

Suppose that G' is the smallest self-conjugate subgroup of G which contains the commutators obtained by using for s and t every pair of non-commutative operators in any system of generators of G. The quotient group of G with respect to G' is generated by its operators that correspond to the given generators of G. The commutators of all these operators must therefore be equal to identity. Hence $G' = G_1$. As a group may generally be generated by a small number of its operators this is a convenient method to find the commutator subgroup of a given group.

For instance, if it is required to find the commutator subgroup of the symmetric group of order 24, we may use for s and t the two operators which may be represented by the substitutions abc and ad respectively. As these two substitutions generate the symmetric group their commutator and its conjugates must generate the commutator subgroup. From

$$s^{-1}t^{-1}st = acb.ad.abc.ad = adb$$

it follows that this subgroup is the alternating group of order 12.

While a group may have a large number of selfconjugate subgroups and even a large number of characteristic subgroups, yet it can have only one commutator subgroup. This is therefore not only a very special selfconjugate sub-

* Jordan : Traité des substitutions, p. 387.
† Cf. Ibid., p. 395.
‡ Cf. Lie : Continuierliche Gruppen, p. 548.

group, but it is also a special characteristic subgroup. It is therefore to be expected that we can prove theorems in regard to it which do not apply to the more general types of subgroups. Some of these have been incidentally noticed above. They may be summarized as follows:

THEOREM I. *When a group (G) is isomorphic to an Abelian group its commutator subgroup (G_1) is of a lower order than G. When this condition is not satisfied it is of the same order and hence identical to G.*

THEOREM II. *The quotient group of G with respect to G_1 is the largest Abelian group to which G has an α, 1 isomorphism.*

THEOREM III. *Every selfconjugate subgroup of G with respect to which it is isomorphic to an Abelian group must include G_1.*

THEOREM IV. *G_1 is a characteristic subgroup of G; i. e., it corresponds to itself in every simple isomorphism of G to itself.*

THEOREM V. *The necessary and sufficient condition that G is solvable is that we arrive at identity by forming the successive derived subgroups of G.*

THEOREM VI. *G_1 is the smallest selfconjugate subgroup of G which contains a commutator of each pair of non-commutative generators in any system of generating operators of G.*

To these theorems we shall add a few which seem to be more special.

THEOREM VII. *If a group of order p^α is not isomorphic to any Abelian group of order p^β it contains only one selfconjugate subgroup of order $p^{\alpha-1}$.*

Since a group of order p^α contains at least one selfconjugate subgroup of order $p^\beta, \beta < \alpha$, the given group must contain at least one selfconjugate of order $p^{\alpha-1}$. With respect to this it is isomorphic to a group of order p^β. All groups of this order are Abelian. As the given group cannot be isomorphic to any Abelian group of a larger order its selfconjugate subgroup of order $p^{\alpha-1}$ must be its commutator subgroup. Hence it contains only one selfconjugate subgroup of this order.

COROLLARY I. *If a group of order p^β contains more than one selfconjugate subgroup of order p it is Abelian.*

COROLLARY II. *Every selfconjugate subgroup whose order is obtained by dividing the order of a group by p or p^2, p being any prime number, includes the commutator subgroup of the group.*

THEOREM VIII. *If a commutator is commutative to one of its two operators its a power is the commutator of the a power of this operator and the first power of the other operator.*

First suppose that $s^{-1}t^{-1}st$ is commutative to t. Then

$$(s^{-1}t^{-1}st)^2 = s^{-1}t^{-1}st\,s^{-1}t^{-1}st = s^{-1}t^{-1}ss^{-1}t^{-1}st^2 = s^{-1}t^{-2}st^2$$

$$(s^{-1}t^{-1}st)^5 = s^{-1}t^{-2}st^2\,s^{-1}t^{-1}st = s^{-1}t^{-2}ss^{-1}t^{-1}st^3 = s^{-1}t^{-3}st^3$$

$$(s^{-1}t^{-1}st)^a = s^{-1}t^{-a+1}st^{a-1}s^{-1}t^{-1}st = s^{-1}t^{-a+1}ss^{-1}t^{-1}st^a$$
$$= s^{-1}t^{-a}st^a.$$

When $s^{-1}t^{-1}st$ is commutative to s we obtain in a similar manner

$$(s^{-1}t^{-1}st)^\beta = s^{-\beta}t^{-1}s^\beta t.$$

Suppose that the operators of G are arranged in rows, each row containing all those that are conjugate to each other and no others; *i. e.*, each row contains one and only one set of conjugate operators. When G is commutative each row consists of a single operator and *vice versa*. All the operators of a row may be obtained by multiplying one of them into certain operators of G. Since all of these last operators are commutators of G the order of G_1 cannot be less than the number of operators that are conjugate to any operator of G.

If we let s represent successively all the operators of a given row while t represents all those of G for each value of s, then will $s^{-1}t^{-1}st = 1$ for g sets of values of s and t. Hence gk of the g^2 commutators of G are equivalent to identity, k being the number of the given rows. By writing after each operator all the different factors which make it equivalent to all the operators of the row in which it occurs, we obtain a system of factors which includes all the different commutators of G. All these factors are commutators and g of them are equal to identity.

It should be observed that the factors which occur after a given operator are conjugate to those which occur after any other operator of the same row. Hence we obtain at least one from each set of conjugate commutators by finding the factors into which we have to multiply one operator from each row in order to obtain all the other operators of the same row. Since the conjugates of a commutator are commutators of the same group, either all or none of the operators of a given row are commutators. This is another proof of the theorem that the commutator subgroup is selfconjugate.*

When one of the given rows contains the square of one of its operators, all its operators are evidently commutators. In general, if one of these rows contains both of the operators s_1 and s_1^a then will s_1^{a-1} and s_1^{1-a} be included among the given factors. Hence these must be commutators of G. From the equation

$$t_1^{-1}s_1t_1 = s_1^a \text{ or } t_1^{-1}s_1t_1s_1^{-1} = s_1^{a-1}$$

* Frobenius, *Sitzungsberichte der Berliner Akademie*, 1896, p. 1348.

we have

$$(t^{-1}s_1t_1s_1^{-1})^\beta = t_1^{-1}s_1^\beta t_1 s_1^{-\beta} = s_1^{(\alpha-1)\beta}$$

Under the given conditions all the powers of $s_1^{\alpha-1}$ must therefore be commutators.

Suppose that G is a transitive substitution group of degree p, p being any prime number. According to Sylow's theorem G contains $kp+1$ (k being some positive integer) conjugate subgroups of order p and its order is ap $(kp+1)$ (a being an integer >0). The number of elements in all its substitutions of order p is $p(p-1)$ $(kp+1)$. This is the total number of elements in all the substitutions of G when $a=1$.* Hence a exceeds 1 whenever k exceeds 0.

The ap substitutions which transform one of the given subgroups of order p into itself form a group whose commutator subgroup is of order 0 or p as a is 1 or greater than 1. If we observe yet that the quotient group with respect to any selfconjugate subgroup of G, except identity, is cyclical and that all the substitutions of order p generate a simple selfconjugate subgroup of G, we obtain the following:

THEOREM IX. *The commutator subgroup of a transitive group of degree p is simple, and it includes all its substitutions of order p when the given transitive group is not regular.*

From this theorem we have directly that a transitive gronp of degree p is solvable when it contains only one subgroup of order p and that it is insolvable whenever it contains more than one such subgroup.

It may be well to add two important theorems in regard to the commutator groups which have been published within the last year, the one by Frobenius and the other by Dedekind. For the proof of these theorems we refer to their articles.

THEOREM X. *The number of linear factors of the group determinate of G is $g \div g_1$.*†

THEOREM XI. *When G is a Hamilton group its commutators are of order 2 or 0.*‡

ANN ARBOR,
July, 1897.

---

* Frobenius, *Crelle*, vol. 101, p. 287.

† *Sitzungsberichte der Berliner Akademie*, p. 1349.

‡ Dedekind, *Mathematische Annalen*, vol. 48, p. 557. The theorem given as a foot note in the same article, p. 553, was previously published, *Quarterly Journal of Mathematics*, vol. 28, p. 266.

ON THE LIMIT OF TRANSITIVITY OF THE MULTIPLY TRANSITIVE SUBSTITUTION GROUPS THAT DO NOT CONTAIN THE ALTERNATING GROUP.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society at the Meeting of December 29, 1897.)

We shall first state a few known principles in the form in which we shall employ them. A k times transitive group (G) involving the following n elements

$$a_1, a_2, a_3, \cdots, a_n$$

contains a simply transitive subgroup (G_1) which is composed of all the substitutions of G that do not contain any one of the following $k-1$ elements

$$a_1, a_2, a_3, \cdots, a_{k-1}$$

It also contains a doubly transitive subgroup (G_2) which is composed of all the substitutions of G that do not contain any one of the following $k-2$ elements

$$a_1, a_2, a_3, \cdots, a_{k-2}$$

etc.

It follows from the definition of a k times transitive group that G contains a subgroup (K_1) which has for one of its transitive constituents the symmetric group of degree k. This constituent cannot be multiply isomorphic to the other constituent of K_1, since G does not include the alternating group of degree n. Hence these two constituents of K_1 have a 1, α isomorphism to each other, α being the number of the substitutions of G that do not contain any one of the following k elements

$$a_1, a_2, a_3, \cdots, a_k$$

Since these α substitutions form a selfconjugate subgroup of the second constituent of K_1, which corresponds to identity in the symmetric group of degree k, the order of K_1 must be $\alpha k!$ while that of G is $\alpha(n-k+1)(n-k+2)\cdots(n-1)n$.

G contains also a subgroup (K_2) which has for one of its transitive constituents the symmetric group of degree $k-1$ and for the other constituent a transitive group which has

an $a(n-k+1)$, 1 isomorphism to this symmetric group. The self-conjugate subgroup of the transitive constituent, corresponding to identity in the symmetric group, is G_1. Similarly, G contains a subgroup (K_3) which has for one of its transitive constituents the symmetric group of degree $k-2$ and for the other an at least doubly transitive group which has an $a(n-k+1)(n-k+2)$, 1 isomorphism to this symmetric group. G_2 corresponds to identity in this isomorphism, etc. The orders of $K_2, K_3, \cdots$ are $a(n-k+1)(k-1)!, a(n-k+1)(n-k+2)(k-2)!, \cdots$.

What precedes includes the following:

THEOREM I. *When a group of degree n is k times transitive there must be an at least β times transitive group of degree $(n-k+\beta)$, $(\beta=1, 2, 3, \cdots, k-2)$, whose quotient group with respect to some selfconjugate subgroup is the symmetric group of degree $k-\beta$.*

COROLLARY. *When no at least a times transitive group of degree n is isomorphic to the symmetric group of degree k a group of degree $n+k$ cannot be more than $k+a-1$ times transitive.*

In order to employ this corollary it is necessary to know the quotient groups of the a times transitive groups of degree n. It is not necessary to consider the alternating or the symmetric group of degree n, since the primitive groups that include such a group are either alternating or symmetric. The simplest case occurs when n is a prime number (p). We shall begin with this case.

All the subgroups of order p that are found in a transitive group of degree p are conjugate. Since such a group is primitive all its selfconjugate subgroups, except identity, are transitive and, therefore, include all the substitutions of order p. If the order of such a selfconjugate subgroup is equal to the order of the group divided by a there are a times as many substitutions of the group that transform a given subgroup of order p into itself as there are substitutions having this property in the selfconjugate subgroup. The quotient group with respect to the given selfconjugate subgroup must, therefore, be isomorphic to the metacyclic group of degree p or to one of its subgroups. Hence we have the

LEMMA I. *The quotient group of any transitive group of degree p with respect to a selfconjugate subgroup that exceeds identity is cyclical.*

If we let $a=1$ and $k=3$ in the given corollary we obtain by means of this lemma that a group of degree $p+3$ cannot be more than three times transitive. Hence we have the

THEOREM II. *A transitive group of degree p cannot occur as a*

subgroup of any primitive group except those of degree p, $p+1$, *and* $p+2$.*

COROLLARY. *A group of degree* $p+k$, $k>2$, *cannot be more than* k *times transitive.*

This important corollary is due to Jordan.†

When a primitive group is of degree $2p$ its order is divisible by p but not by p^2; for, if its order were divisible by p^2 it would contain a commutative subgroup of this order and hence it would have to contain a transitive subgroup of degree p. All its subgroups of order p must therefore be conjugate and all its substitutions of this order must be included in every selfconjugate subgroup, except identity. Hence every quotient group with respect to a selfconjugate subgroup whose order exceeds unity is isomorphic to the subgroup which includes all the substitutions that transform a given subgroup of order p into itself.

Since every simple isomorphism of a metacyclic group, or one of its subgroups, to itself can be obtained by transforming it by means of given substitutions, it follows that the quotient group, with respect to a selfconjugate subgroup whose order exceeds 2, of the given subgroup that contains all the substitutions of the primitive group which transform the cyclical subgroup of order p into itself is commutative. It is either cyclical or contains a cyclical subgroup of half its order. Hence we obtain the following

LEMMA II. *The quotient group of any primitive group of degree* $2p$ *with respect to a selfconjugate subgroup whose order exceeds unity is commutative. It is either cyclical or it contains a cyclical subgroup of half its order.*

We consider now the nonprimitive groups of degree $2p$ which contain two systems of nonprimitivity without containing also p such systems and whose orders are not divisible by p^2. The subgroup which includes all the substitutions that do not interchange the systems of nonprimitivity is formed by making a transitive group of degree p simply isomorphic to itself. Hence every selfconjugate subgroup whose order exceeds unity includes all the substitutions of order p. In exactly the same way as we pursued above we obtain the following:

LEMMA III. *The quotient group, with respect to any selfconjugate subgroup except unity, of any nonprimitive group of degree* $2p$ *which contains two systems of nonprimitivity without containing also* p *such systems and whose order is not divisible by* p^2 *is com-*

* Such a transitive group occurs in every symmetric or alternating group whose degree exceeds $p-1$, but we do not consider these groups in this paper

† *Bulletin de la Société Mathématique de France*, vol. 1, p. 41.

mutative. It is either cyclical or it contains a cyclical subgroup of half its order.

The nonprimitive groups of degree $2p$ which contain p systems of nonprimitivity are isomorphic to transitive groups of degree p. The selfconjugate subgroup which corresponds to identity in this isomorphism is of order 2^a. It is easy to see that the quotient group of such a nonprimitive group cannot be the symmetric group of order 6 when $p > 3$; for if we take a quotient group with respect to a selfconjugate subgroup that includes all of the given 2^a substitutions it is either a cyclical group or a transitive group of degree p. If the given selfconjugate subgroup includes only half of the given 2^a substitutions the quotient group contains a selfconjugate subgroup of order 2, and if it includes less than half the order of the quotient group is divisible by 4. Hence we have

LEMMA IV. *A transitive group of degree* $2p$, $p > 3$, *which does not contain any regular substitution of degree* p *cannot be isomorphic to the symmetric group of order* 6.

Letting $a = 1$ and $k = 3$ in the corollary under theorem I we observe by means of this lemma and the fact that no group of degree 7 or 9 is more than 3 times transitive; that a group of degree $2p + 3$ cannot be more than 3 times transitive. Hence we have the

THEOREM III. *A group of degree* $2p + k$, $k > 2$, *cannot be more than* k *times transitive.**

It may happen that this theorem will give a lower limit of transitivity than the corollary under theorem II., *e. g.*, $17 = 2 \times 7 + 3$ or $13 + 4$. Hence a group of this degree cannot be more than 3 times transitive, according to this theorem and no more than 4 times according to the given corollary. On the contrary, $100 = 97 + 3$ or $2 \times 47 + 6$. Hence a group of degree 100 cannot be more than 3 times transitive according to the given corollary and no more than 6 times according to this theorem.

It may be observed that we have used only K_2 in seeking for these limits of transitivity. K_3, K_4, $\cdots$ are, in general, not so suitable to find low limits, since the corresponding symmetric groups are of lower degrees. As the degree of K_1 is not always equal to $n - k$, it also does not appear to be as suitable for this purpose as K_2. For low limits of transitivity of the groups of a general degree we must refer to the articles of Jordan and Bochert in recent volumes of *Liouville's Journal* and *Mathematische Annalen* respectively.

CHICAGO, *October*, 1897.

* In the corresponding theorem Jordan assumes that $k > 3$ when p is of the form $3m + 1$, loc. cit.

GEOMETRY OF SOME DIFFERENTIAL EXPRESSIONS IN HEXASPHERICAL COÖRDINATES.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society at its Fourth Summer Meeting, Toronto, Canada, August 16, 1897.)

THIS paper is to be regarded as an appendix to my dissertation: Ueber die linearen Complexe der Lie'schen Kugelgeometrie (Göttingen, Kaestner, 1895).

It will simply give an outline of differential geometry, and show its application to the quadratic complex.

Let $x_1, x_2 \cdots x_6$ be any six variables satisfying a homogeneous quadratic identity; these variables may be regarded as the six homogeneous coördinates of the sphere. The form here assumed will be

$$\Pi(x) \equiv \sum_{i=1}^{6} x_i^2 = 0, \tag{1}$$

where two of the coördinates must be imaginary to represent a real sphere.

The geometric meaning of the variables x_i is essentially the same as the $\xi, \eta, \zeta \cdots$ in my classification of Dupin's Cyclides,* from which they can be derived by a linear transformation.

A homogeneous equation of degree n among the variables $x_1 \cdots x_6$,

$$f(x) = 0. \tag{2}$$

in connection with (1) will define a spherical complex of degree n.

In the neighborhood of any given sphere x', belonging to the complex, it can be replaced by a linear complex

$$\sum_{i=1}^{6} \frac{\partial f}{\partial x_i'} y_i = 0, \tag{3}$$

where y_i are running coördinates.

This complex is called a linear tangent complex to $f(x)$ at x'; every sphere has a corresponding tangent complex, and in fact a whole pencil of them. $f(x) = 0$ is not changed by adding any multiple of $\Pi(x)$ to it, hence

* Criteria for nodes in Dupin's cyclides, with a corresponding classification, *Annals of Mathematics*, vol. 11, No. 5, p. 137, June, 1897.

$$(4) \qquad \sum_{i=1}^{6} \left(\lambda \frac{\partial f}{\partial x_i'} + \mu x_i' \right) y_i = 0$$

will represent such a complex for every value of $\lambda : \mu$. This pencil has a linear congruence of spheres in common; its directrices are defined by

$$(5) \qquad \lambda^2 \sum_{i=1}^{6} \left(\frac{\partial f}{\partial x_i'} \right)^2 + 2\lambda\mu \sum_{i=1}^{6} x_i' \frac{\partial f}{\partial x_i'} + \mu^2 \Pi(x_i') = 0$$

but as $\Pi(x)$ is identically zero, and

$$\sum_{i=1}^{6} x_i' \frac{\partial f}{\partial x_i'} = 0 \text{ when } f(x') = 0,$$

the congruence is special when

$$\sum_{i=1}^{6} \left(\frac{\partial f}{\partial x_i'} \right)^2 + 0,$$

and degraded when

$$\sum_{i=1}^{6} \left(\frac{\partial f}{\partial x_i'} \right)^2 = 0.$$

In the former case the congruence is composed of those spheres which touch x' in its circle of intersection with the fundamental sphere of (3), to which it belongs.

This circle is called the *trajectory circle* of x'; every sphere of the complex has a trajectory circle, and spheres of the complex that are infinitesimally close to a given sphere and touch it, must touch it in points of its trajectory circle. *

The expression

$$(6) \qquad \sum_{i=1}^{6} \left(\frac{\partial f}{\partial x_i} \right)^2$$

is a differential invariant † and has an important geometric meaning. When it vanishes, (3) is a special complex and the congruence defined by (4) becomes degraded. The trajectory circle of such a sphere reduces to a point, and the sphere itself is called *singular*. There are ∞^2 spheres fulfilling the conditions

$$f(x) = 0, \Sigma \left(\frac{\partial f}{\partial x} \right)^2 = 0,$$

which envelop a surface called the *surface of singularities* of $f(x) = 0$. ‡

* Cf. Lie, in *Mathematische Annalen*, vol. 5, p. 207.
† See *e. g.* Koenigs: La géométrie reglée, p. 76.
‡ Klein: Höhere Geometrie, vol. 1, p. 485.

When (6) vanishes identically, when $f(x) = 0, \prod(x) = 0$, then every directrix of the degraded congruence (4) belongs to $f(x)$ * and every sphere of $f(x)$ is singular; the complex consists of ∞^2 tangent pencils of spheres, and their points of tangency envelope a surface. Such a complex is called *special*, and may be said to have an envelope.

Consider the two complexes of orders n, m,

$$f(x) = 0, \ \varphi(x) = 0.$$

Let x' be a sphere of the congruence defined by these complexes; the congruence can be replaced in the neighborhood of x' by the system of ∞^2 linear complexes or the ∞^3 linear congruences,

$$\Sigma\left(\lambda\frac{\partial f}{\partial x_i'} + \mu x_i'\right) y_i = 0$$

$$\Sigma\left(\rho\frac{\partial \varphi}{\partial x_i'} + \sigma x_i'\right) y_i = 0.$$

These congruences have a linear series in common, defined by the equations

$$(7) \quad \Sigma\frac{\partial f}{\partial x_i'} y_i = 0; \qquad \sum_{i=1}^{6}\frac{\partial \varphi}{\partial x_i'}\cdot y_i = 0; \qquad \sum_{i=1}^{6} x_i' y_i = 0,$$

but since the combinant of these forms vanishes identically, the envelope of the series is a gauche quadrilateral of minimum lines.

Spheres of the linear series break up, therefore into two tangent pencils; x' is the only sphere belonging to both. As x' varies, the points of intersection of two pairs of these minimum lines generate the two mantles of a surface, the *focal surface* of the congruence. *A congruence of spheres is usually composed of the spheres which are doubly tangent to a surface.*

If all the first minors of the combinant of (7) vanish, the two tangent pencils coincide; that is, for such values of x' that satisfy

$$(8) \quad \cdot \ \Sigma\left(\frac{\partial f}{\partial x_i}\right)^2 \cdot \Sigma\left(\frac{\partial \varphi}{\partial x_i}\right)^2 - \Sigma\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial \varphi}{\partial x_i}\right) = 0.$$

Such spheres have a contact of the second order with the focal surface; they are therefore principal spheres.

* Klein: Differentialgleichungen in der Liniengeometrie, *Mathematische Annalen*, vol. 5, p. 288.

If (8) is identically satisfied by all the spheres of the given congruence, the congruence is composed of the principal spheres of one generation, belonging to the focal surface.

If one of the complexes, φ, be not given, (8) will represent the differential equation of all the congruences of principal spheres contained in f.

The principal spheres of a surface cannot belong to a linear complex, for adjacent spheres must touch each other. If two adjacent spheres of a linear complex touch, the whole pencil must belong to the complex.

Consider the congruence formed by a special complex, and a linear one. If (8) is not satisfied, the congruence must also envelope another surface; if (8) is satisfied, the congruence consists of the ∞^1 tangent pencils which touch the given surface in the points of the curve of intersection with the fundamental sphere of the linear complex.

The spheres of the congruence cut the sphere at a constant angle; hence the curve of intersection, being a line of curvature on the sphere, must be a line of curvature on the surface.* Hence

Among the ∞^3 spheres which touch a given surface, there are ∞^2 which also cut a fixed sphere at a constant angle. These spheres will either envelope another surface or be arranged in ∞^1 pencils, touching the surface along the curve of intersection with the sphere, which is then a line of curvature of the given surface.†

The points of space form a linear complex; the point-spheres belonging to a general complex lie on a surface. The degree of this surface may be found as follows: take any two linear complexes which contain the plane at infinity; their fundamental spheres are planes, which are likewise the surfaces of point-spheres of the complexes. Now find the number of spheres the given complex, the two linear complexes and the complex of points have in common—this gives $2n$; so an arbitrary line pierces the surface in $2n$ points. Further, by taking two complexes not containing the plane at infinity; the point spheres common to the congruence lies on a circle—by the same reasoning, an arbitrary circle will intersect the surface in $2n$ points, which necessitates that the surface contains the circle at infinity as an n fold line, hence:

The locus of the point-sphere in a spherical complex of degree n

* Cf., *e. g.*, Knoblauch: Theorie der krummen Flächen, p. 263.

† Darboux, in his Théorie des Surfaces, vol. 1, p. 257, states the general theorem, having derived it by means of inversion of planes; he does not mention the exceptional case.

is a surface of degree $2n$, and containing the circle at infinity as an n fold line.

The planes of space form a linear complex, the planes contained in a general complex lie on a surface of class $2n$.

When

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i}\right)^2=0,$$

these two surfaces must be identical.

The spheres common to three complexes, $f(x)=0$, $\varphi(x)=0$, $\chi(x)=0$ of order n, m, p form a series; they may envelope an annular surface, or a curve, or simply be contained in a hyper pencil, all the spheres containing a minimum line of one generation in common.

At any sphere x', the series can be replaced by ∞^1 Dupin's cyclides (linear series), all of which have two coincident spheres in common.

If the first minors of the vanishing combinant do not identically vanish, the series envelopes an annular surface, in which $3mnp(m+n+p-3)$ spheres are touched by the consecutive one. Finally if the first minors vanish identically, the series envelopes a curve or is contained in a hyperpencil, as every sphere touches the consecutive one. The determinant expressing this condition may be regarded as the differential equation of a line of curvature on the focal surface of every congruence satisfying it.

A series contained in the complex of points defines the focal lines of the focal surface of the congruence. The series common to a general complex and the congruence of points and planes defines the singular line of curvature, which is also a minimum curve on the locus of points contained in the general complex.

Many interesting properties of annular surfaces may be obtained by regarding the coördinates of a sphere, $x_1 \cdots x_6$ as functions of a new variable t; the corresponding study has been made of ruled surfaces by Koenigs, in his Géometrie Réglée, and all his results may be interpreted by spheres.

Application to the quadratic complex.

Let $n=2$. A general quadratic equation in 6 variables can be transformed by a linear transformation to a sum of squares, as

$$f(x)=\sum_{i=1}^{6}a_ix_i^2=0,$$

and the quadratic identity simultaneously to the form

$$\Pi(x) = \sum_{i=1}^{6} x_i^2 = 0.*$$

The singular spheres will be determined by those values for x_i which satisfy $f = 0$, $\Pi = 0$, and

$$\sum_{i=1}^{6} \left(\frac{\partial f}{\partial x_i} \right)^2 \equiv \sum_{i=1}^{6} a_i^2 x_i^2 = 0.$$

Let $\frac{\partial f}{\partial x_i}$ be denoted by z_i; z_i are now the coördinates of a sphere which touches x (x a singular sphere) in its trajectory circle which is a point. z is also a tangent to the surface of singularities at the same point as x; any sphere of the tangent pencil is therefore expressible in the form

$$\lambda x + z$$

where λ is a parameter. When x successively takes the value of every singular sphere, this expression will represent all the ∞^3 tangent spheres to the surface of singularities.

Consider the equations

$$\Sigma x_i^2 = 0; \quad \Sigma a_i x_i^2 = 0; \quad \Sigma a_i^2 x_i^2 = 0.$$

The first one expresses that x is a sphere, the second, that it belongs to the given complex, and the third, that it is a singular sphere. Let a tangent of the surface of singularities be donoted by m,

$$m_i = \lambda x_i + z_i = \lambda x_i + a_i x_i.$$

Substitute these values for x in the equations

$$\Sigma m_i^2 = 0; \quad \Sigma x_i m_i = 0; \quad \Sigma x_i^2 = 0$$

that express that x_i, m are spheres of a tangent pencil. Hence

$$\sum_{i=1}^{6} \frac{m_i^2}{a_i + \lambda} = 0, \quad \Sigma \frac{m_i^2}{(a_i + \lambda)^2} = 0, \quad \Sigma m_i^2 = 0.$$

When λ is eliminated between these equations, that is, the λ discriminant of the first one is equated to zero, it will represent the surface of singularities in tangential coördinates. This equation is of degree 4 in λ, its discriminant is, therefore of degree 12 in m, $\phi(m) = 0$.

* For the general discussion, see Weierstrass: Zur Theorie der bilinearen und quadratischen Formen, *Berliner Monatsberichte*, 1868, and for its application to 6 variables, cf. Klein, in *Mathematische Annalen*, vol. 23, p. 538, and Weiler, in *Mathematische Annalen*, vol. 7, p. 145. Only the general form will be considered here.

Let y, t be any two spheres not in contact; $y + kt$ will be a sphere containing their circle of intersection.

$\varphi(y + kt) = 0$ is an equation of degree 12 in k; through any given circle 12 spheres can be passed, which touch the surface. Especially, when y, z are planes, $y + kt$ will represent a book of planes containing their line of intersection. From this it follows that *the surface of singularities of the quadratic spherical complex is of class* 12.*

The further study of the complex will be facilitated by comparing each theorem with the corresponding theorem in line geometry. All of these results are known, but it is desirable to show how each system can be derived from the other, independent of the methods of Darboux (l. c.), Loria† Bôcher,‡ and the smaller works of Reye, Moutard, Laguerre, Maxwell, Casey, etc.

The relation between the two spaces is that discussed by Lie§ and the notation employed will be that of Klein, in his manuscript lectures on Liniengeometrie.

Suppose $x_6 = 0$ is the equation of the point complex; it is taken as one of the complexes of reference.

The six equations $x_i = 0$ represent six general linear complexes, mutually in involution; one of them is the complex of points, hence being in involution necessitates their being orthogonal, and the fundamental sphere of each belongs to each of the others (except x_6 whose fundamental sphere is entirely illusory). This gives five mutually orthogonal spheres, discussed by Darboux in his chapter on Coordonnées Pentaspheriques. (Théorie des Surfaces, vol. 1.) These linear complexes, together with their congruences and series have been exhaustively studied by Domsch.‖

Now consider the series of quadratic complexes

$$\sum_{i=1}^{6} \frac{x_i^2}{a_i + \lambda} = 0. \tag{9}$$

From the method in which it was derived it is at once evident that the whole set has a common surface of singularities; this surface is the same one as was discussed above,

* Darboux, in his "Sur une classe remarquable" obtains this theorem in practically this way, though he does not define the surface as *surface of singularities*. Cf. p. 276.

† Geometria della sfera, *Memorie di Torino*, II., 36, 1884.

‡ Reihenentwickelungen in der Potentialtheorie, Göttinger Preisschrift, 1891.

§ First discussed in *Mathematische Annalen*, vol. 5, and more in detail in his Geometrie der Berührungstransformationen, pp. 444–475.

‖ Darstellung der Flächen 4ter Grades mit einem Doppeltkegelschnitt durch hyperelliptische Reihen, *Grunert's Archiv*, 2d Series, Part 2, 1885.

for the original complex $\Sigma a_i x_i^2 = 0$ belongs to this set, corresponding to $\lambda = \infty$. Clearing the equation of fractions,

$$\lambda^5 \Sigma x_i^2 + \lambda^4 [(\Sigma a_i)(\Sigma x_i^2) - \Sigma a_i x_i^2] + \cdots = 0,$$

but $\Sigma x_i^2 \equiv 0$, and if the quartic has as an infinite root, $\Sigma a_i x_i^2 = 0$, which is the equation of the original complex.

If x_i be the coördinates of any sphere, in general 4 corresponding values of λ can be found, *i. e.*, any sphere of space belongs to four complexes of this set; when λ is a double root of the quartic equation, the corresponding sphere is a singular sphere of the corresponding complex, as was shown above. It then belongs to two other complexes, but not as singular sphere.

In every tangent pencil of spheres of the surface of singularities is a singular sphere corresponding to each value of λ, hence the ∞^3 tangent spheres are the singular spheres of the set of ∞^1 complexes

$$\sum_{i=1}^{6} \frac{x_i^2}{a_i + \lambda} = 0.$$

In a quadratic line complex, a singular line lies in a plane, whose complex-conic breaks up into two pencils. The surface of singularities is a Kummer surface; it is of the fourth order and fourth class. The vertices of the pencils are the points where the singular tangent line cuts the surface again. When the singular line is an inflexional tangent, three of its points of intersection coincide, hence in this case the whole pencil belongs to the quadratic complex. At every point there are two inflexional tangents, hence there are two complexes of the set which contain the entire pencil of tangents at that point. These lines are transformed, in spherical geometry, into the two principal spheres; the two complexes which contain one or the other of the two principal spheres contain the entire pencil of spheres at that point. This explains why any singular sphere belongs to two other complexes of the set, not as singular sphere. In case of a principal sphere, however, it can only belong to one other complex, the one containing the other principal sphere at that point as singular sphere.

This requires that λ is a triple root of (9); hence *the principal spheres of the surface of singularities of the quadratic complexes* (9) *are defined by the equations*

$$\sum_{i=1}^{6} \frac{x_i^2}{a_i + \lambda} = 0, \quad \sum_{i=1}^{6} \frac{x_i^2}{(a_i + \lambda)^2} = 0, \quad \sum_{i=1}^{6} \frac{x_i^2}{(a_i + \lambda)^3} = 0.$$

By giving λ a fixed value, these equations define the series of principal spheres contained in the corresponding complex.

In line geometry, the inflexional tangents envelop a curve of order 16 on the Kummer surface, which is an asymptotic line.* The principal spheres define a line of curvature on the surface. By varying λ all of the lines of curvature can be obtained.

Among the complexes of the set (9) are six, corresponding to $\lambda = -a_\nu$, which degrade to linear complexes counted twice. The singular spheres of such complexes are those that belong to the contiguous complex of the set, which is found to be

$$\sum_{i=1}^{6} \frac{x_i}{a_i - a_\nu} = 0. \quad [i + \nu].$$

This defines 6 congruences

$$x_\nu = 0 \sum_{i=1}^{6} \frac{x_i^2}{a_i - a_\nu} = 0 \quad [i + \nu]$$

which have the surface of singularities as their common focal surface.

In line geometry, at every point of tangency on a Kummer surface, are six tangents which touch the surface again; these lines are lines of the tangent pencil which belong to the six complexes $x_\nu = 0$.

The focal surface of

$$\sum_{i=1}^{5} \frac{x_i^2}{a_i - a_6} = 0, \; x_6 = 0$$

is the surface of points in the complex

$$\sum_{i=1}^{5} \frac{x_i^2}{a_i - a_6} = 0$$

and, as was shown above, is a surface of order 4, containing the circle at infinity as a double line. This surface was defined by Darboux† as a *cyclide*, hence

The surface of singularities of a quadratic spherical complex is a cyclide.

There are five systems of spheres which are doubly tangent to the surface; by reducing to point coördinates, their centres are found to lie on quadratic surfaces, and, as was

*Klein and Lie, *Berliner Monatsberichte*, 1870, reprinted in *Mathematische Annalen*, vol. 23, p. 579.

† Mémoire sur une classe de courbes et de surfaces, *Comptes rendus*, vol. 68, p. 1311.

shown above, those of each system cut a fixed sphere orthogonally. This is exactly the method of generation given by Darboux (*l. c*) (and expanded in his monogram of the same name, 1873).

Each sphere cuts the cyclides along a curve of the fourth order with two double points; hence the curve breaks up into two circles; hence, *on the cyclide are ten series of circles.* This surface is therefore included among those studied by Kummer,* that have one or more series of conics upon them.

The Kummer surface contains 16 double points, and 16 double planes; six of the double points lie in each of the double planes, and the same, dually. Hence upon the surface of the cyclide are 16 minimum lines, each of which will cut five others.

In each of the line congruences

$$x_\nu = 0, \quad \sum_{i=1}^{6} \frac{x_i^2}{a_i - a_\nu} = 0 \quad [i \neq \nu].$$

are ∞^1 lines whose two points of tangency on the Kummer surface become coincident so that the lines cut the surface in 4 coincident points; in this, as the general case, the whole tangent pencil belongs to the linear complex $x_\nu = 0$. The locus of these points is an asymptotic line of the order 8; there are six such curves on the surface.

The congruence

$$x_6 = 0, \quad \sum_{i=1}^{5} \frac{x_i^2}{a_i - a_6} = 0$$

becomes the line of curvature on the cyclide which is likewise a minimum curve. It is of order 8.

As the tangent planes at the points of each of the special asymptotic curves belong to the linear complex, the corresponding spherical congruence will contain a tangent pencil of spheres; this pencil will contain one point sphere—hence the curve is the curve of intersection of the cyclide and the five orthogonal spheres. These are the focal lines of the cyclide; they are of order 4, and are the curves of contact of the minimum developable circumscribed about the cyclide and the circle at infinity.

Finally, consider the ∞^1 congruences defined by

$$x_6 = 0, \quad \sum_{i=1}^{5} \frac{x_i^2}{a_i - \lambda} = 0,$$

* Ueber die Flächen vierten Grades auf welche Schaaren von Kegelschnitten liegen, *Berliner Monatsberichte*, 1863.

where λ is given, in succession, every value. The focal surfaces will be a system of ∞^1 Kummer surfaces, which touch each other along the six special asymptotic curves of order 8. *

An arbitrary line of $x_6 = 0$ belongs to three of the congruences; its three pairs of points of tangency define a double involution, *i. e.*, every pair is harmonic with the other two. †

Five of the congruences are linear, corresponding to $\lambda = a_1, a_2, \cdots a_5$; their focal surfaces are five pairs of straight lines, each pair being conjugate with regard to x_6.

This gives, in spherical geometry, ∞^1 quadratic congruences, contained in the complex of point spheres; *i. e.*, ∞^1 cyclides having common focal lines. Through each point pass three surfaces of the system. The six points on a tangent to three Kummer surfaces become six minimum lines through a point; each pair determines a tangent plane to a cyclide through a point; these planes make conjugate traces on the circle at infinity, hence the three cyclides passing through any point intersect each other orthogonally. Their curves of intersection are therefore lines of curvature and are curves of the 8 order.

Five of the cyclides reduce to spheres, the fundamental spheres of $x_1 = 0, \cdots x_5 = 0$. The surface of singularities of the complexes (9) is included in this system of confocal cyclides, corresponding to $\lambda = a_6$.

When

$$\sum_{i=1}^{6} \left(\frac{\partial f}{\partial x_i} \right)^2 = \sum_{i=1}^{6} a_i^2 x_i^2$$

is identically zero, the surface of singularities in line geometry becomes a quadric surface; in spherical geometry it coincides with the point locus and envelope of planes contained in the complex—hence a surface of the fourth order and fourth class; it is a Dupin's cyclide; hence

The Dupin cyclide is the only surface that can be the complete envelope of a non reducible special quadratic spherical complex.

CORNELL UNIVERSITY,
August 12, 1897.

* This remark is made by Lie in *Mathematische Annalen*, vol. 5, p. 255, in a foot-note, inferring it from the results of spherical geometry. It is proved independently by Reye in his memoir Ueber die Singularitätenflächen quadratischer Strahlencomplexe und ihre Haupttangentencurven, *Crelle's Journal*, vol. 97, p. 247.

† Klein, in Linien- und Metrische Geometrie, *Mathematische Annalen*, vol. 5.

LIE'S DIFFERENTIAL EQUATIONS.*

SOPHUS LIE—*Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen.* Bearbeitet und herausgegeben von DR. GEORG SCHEFFERS. Leipsic, Teubner. 1891. 8vo, pp. xiv + 568.

These lectures constitute one of the courses of Lie's cycle which has been repeating itself at the University of Leipsic since 1886. The course serves the double purpose of an elementary introduction to the theory of continuous groups and an exposition of how that theory subordinates the various heterogeneous methods of integrating ordinary differential equations to one general method, the key to which is the notion of an infinitesimal transformation, first introduced by Lie at the inception of his theories. The lectures have been edited with the double object of both scientific and pedagogic usefulness. They are so designed that a fourth semester student of a German university is prepared to read them, and they should offer no difficulty to the American reader who is familiar with the processes of the infinitesimal calculus. The numerous problems and illustrative examples drawn from geometry and mechanics commend the book to the private student.

The book falls into five parts: I. The Notions—Infinitesimal Transformation and One Parameter Group of the Plane, chapters 1–4, pp. 1–85; II. Utility of the Notion of Infinitesimal Transformation in Differential Equations of the First Order in Two Variables, chapters 5–9, pp. 86–187; III. One Parameter Groups in Three Variables, chapters 10–13, pp. 187–286; IV. One Parameter Groups and Infinitesimal Transformations in n Variables, Application of these Notions to Differential Equations, chapters 14–20, pp. 286–472; V. Integration of Ordinary Differential Equations of the Second Order which Admit of a One Parameter Group, and Related Problems, chapters 21–25, pp. 473–566.

I. A point transformation is an operation by which a point is carried into the position of a point. Two equations of the form

$$x_1 = \varphi(x, y), \quad y_1 = \psi(x, y), \quad \frac{\partial(\varphi, \psi)}{\partial(x, y)} + 0, \qquad (1)$$

φ and ψ being regular analytic functions, are said to determine a point transformation of the plane into itself. If the equations (1) contain a parameter a, they define a family of

* An interesting account of this work, from a somewhat different point of view, was contributed by Professor E. Study to the *Zeitschrift für Mathematik und Physik*, vol. 38 (1893), pp. 185–192.—EDITORS.

∞^1 such transformations. Such a family of ∞^1 transformations constitutes a one parameter finite continuous group of transformations when the successive performance of any two transformations of the family is equivalent to a transformation belonging to the family, *i. e.*, if the ∞^1 transformations

$$x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a) \tag{2}$$

form a continuous group, the elimination of (x_1, y_1) between these equations and the two following

$$x_2 = \varphi(x_1, y_1, a_1), \quad y_2 = \psi(x_1, y_1, a_1),$$

must give rise to

$$x_2 = \varphi(x, y, a), \quad y_2 = \psi(x, y, a),$$

where a is a function of a and a_1 alone.

Only those groups are studied which contain the inverse transformation of every transformation in them, *i. e.*, whose transformations can be arranged in pairs as inverse.* Accordingly, to every value of the parameter a there corresponds a value $\bar{a}$ such that

$$x = \varphi(x_1, y_1, \bar{a}), \quad y = \psi(x_1, y_1, \bar{a}).$$

Since the successive performance of a transformation and its inverse yields the identical transformation, there is some value of the parameter a, say a_0, which gives the identical transformation

$$x = \varphi(x, y, a_0), \quad y = \psi(x, y, a_0),$$

leaving all points at rest. A value of the parameter differing by an infinitesimal from that of the parameter of the identical transformation gives the transformation

$$x_1 = \varphi(x, y, a_0 + \delta a) = x + \xi(x, y)\,\delta t + \cdots,$$
$$y_1 = \psi(x, y, a_0 + \delta a) = y + \eta(x, y)\,\delta t + \cdots,$$

by which the point (x, y) is transformed into a point $(x + \delta x, y + \delta y)$ infinitely near, where

$$\delta x = \xi(x, y)\,\delta t, \quad \delta y = \eta(x, y)\,\delta t,$$

to terms of the second order. Such a transformation is called an infinitesimal transformation.

Consider now several examples of ordinary differential equations that are readily integrable. The differential

* This limitation is really no restriction. It is only formal since Lie has devised means for deriving the defining equations of any finite continuous group from those of a group whose transformations are inverse in pairs. See Theorie der Transformationsgruppen, vol. 3, theorem 26.

equation $y' = f(x + ky)$ assigns the same direction y' to every point of the straight line

$$x + ky = constant. \tag{3}$$

Hence all the integral curves of this equation may be derived from any one of them by the translation of that one along the parallel straight lines (3). These translations however form a one-parameter group defined by the equations

$$x_1 = x - ka, \quad y_1 = y + a.$$

By the transformations of this group the integral curves of the above differential equation are changed one into another.

The homogeneous equation

$$y' = f\left(\frac{y}{x}\right)$$

gives to every point of the straight line $y = mx$ the same direction; hence the integral curves of the homogeneous differential equation can be found from any one of them by proportionately increasing or diminishing the latter from or toward the origin. These proportionate increasings and diminishings from the origin out constitute a group of operations, namely the one parameter group of so-called similitudinous transformations, $x_1 = ax$, $y_1 = ay$. Here again the family of integral curves as a whole remains invariant, the curves of the family being interchanged among themselves when all the transformations of the one parameter group are performed.

As a third example the integrable form

$$xy' - y - (x + yy')f(x^2 + y^2) = 0$$

assigns to the points of a circle with centre at the origin directions which make the same constant angle with the circle, as is shown by the equivalent form

$$\left(y' - \frac{y}{x}\right) \Big/ \left(1 + y'\frac{y}{x}\right) = f(x^2 + y^2),$$

where the left hand member represents the cotangent of the angle in question. Hence all the integral curves cut the circle under the same angle and accordingly they may all be found by rotating any one of them about the origin. All these rotations about the origin make up the one parameter group, $x_1 = x\cos\alpha - y\sin\alpha$, $y_1 = x\sin\alpha + y\cos\alpha$.

It will also be observed that in any one of these illustrations an infinitesimal transformation of the group considered

changes integral curve into an integral curve. In each case the differential equation or the family of integral curves is said to admit of the one parameter group of transformations. These examples show that when the integration of a differential equation can be effected there can be given a group of point transformations which transform the points of an integral curve into those of an integral curve. This fact suggests the high probability that the knowledge of such a group of transformations may be used to simplify and indeed methodically to systematize the processes of integration. It is largely the business of the sequel of the book before us to reduce this probability to a certainty and thereby to develop a general method of integration of ordinary differential equations on the connection between differential equation and group of transformations pointed out in the above simple examples. This general method is the outgrowth of Lie's discoveries made in the years 1869 to '74.

The intimate relationship between the notions one parameter group and infinitesimal transformation is of prime importance. In fact from Lie's theorem that a one parameter group contains but one infinitesimal transformation and conversely that an infinitesimal transformation carried out successively generates but one one parameter group,* the two notions may be regarded as coextensive.

Two infinitesimal transformations whose ξ and η differ by the same constant factor are said to be dependent; they are in no way essentially different, since δt is arbitrary. The finite equations of the group generated by the infinitesimal transformation are found by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)} = dt,$$

with the initial conditions

$$x_1 = x, \quad y_1 = y, \quad t = 0,$$

in the form

$$\Omega(x_1, y_1) = \Omega(x, y), \quad W(x_1, y_1) - t = W(x, y), \qquad (4)$$

or solved with regard to x_1, y_1 and developed in powers of t,

$$x_1 = x + \xi(x, y)\, t + (\xi\xi_x + \eta\,\xi_y)\,\frac{t^2}{1\cdot 2} + \cdots,$$

* It is to be noted that when the term group is used in this review a one-parameter group of transformations inverse in pairs is meant. Similarly transformation in any geometrical connection is short for point transformation.

$$y_1 = y + \eta(x,y)\,t + (\xi\eta_x + \eta\,\eta_y)\frac{t^2}{1\cdot 2} + \cdots .$$

The infinitesimal transformation appears again by putting δt for t.

If we remark that a change of variables does not affect the group property, by the substitution

$$\Omega(x,y) = \bar{x}, \quad W(x,y) = \bar{y},$$

the above equations (4) assume the form

$$\bar{x}_1 = \bar{x}, \quad \bar{y}_1 = \bar{y} + t;$$

hence every one parameter group can be brought to the form of a group of translations by a proper change of variables. This form Lie calls the canonical form of the group and the reducing variables its canonical variables.

Lie adopts

$$Uf \equiv \xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} \equiv \xi p + \eta q$$

as the symbol of an infinitesimal transformation. This choice of a symbol, which, by the theorem quoted above, may represent the infinitesimal transformation or the one-parameter group indifferently, is a peculiarly happy one, because of several of its properties: 1° $Uf\delta t$ is the increment which an arbitrary function $f(x, y)$ receives by the transformation; 2° an arbitrary function $\varphi(x_1, y_1)$ is given by the series

$$\varphi(x_1\,y_1) = \varphi(x,y) + \frac{t}{1}U\varphi(x,y) + \frac{t^2}{1\,2}UU\varphi(x,y) + \cdots;$$

3° the form of Uf is unchanged when new variables, functions of the old, are introduced; 4° the commutator* of any two Uf's is a symbol of the same form, namely,

$$\begin{aligned}(U_1U_2)f &\equiv U_1(U_2f) - U_2(U_1f)\\ &\equiv (U_1\xi_2 - U_2\xi_1)p + (U_1\eta_2 - U_2\eta_1)q;\end{aligned}$$

5° the symbol of an infinitesimal transformation of an r parameter group is expressible linearly with constant coefficients in symbols of the above form; 6° it is a convenient

* This may perhaps be taken as an English equivalent of *Klammerausdruck*; it is suggested by the fact that the vanishing of the expression is the condition that the two operations be commutative. A direct translation of the term would be very cumbersome.

representative of a group when the group becomes the object of another group. All necessary computations are made with this symbol. This operator U is to the theory of continuous groups what the operation of differentiation is to the calculus.

Since the symbol Uf preserves its form when new variables are introduced every one parameter group may be changed into every other. If $x'(x, y)$ and $y'(x, y)$ are the new variables,

$$U'f = Ux' \frac{\partial f}{\partial x'} + Uy' \frac{\partial f}{\partial y'};$$

hence the solutions of $Ux' = 0$ and $Uy' = 1$ give the canonical variables of the group Uf; these canonical variables are accordingly determined by an integration followed by a quadrature.

One of the first questions to arise in studying a family or group of transformations is—What functions, equations and geometrical configurations are invariant by the transformations? If the transformations do not form a continuous group they may or may not have invariants;* if they form a continuous group the group must have invariants. At this point Lie establishes the beautiful theorem—In order that a function, equation or curve admit of all the finite transformations of a one parameter group it is necessary and sufficient that it admit of the infinitesimal transformation of the group.

The invariant functions are the solutions of the partial differential equation $Uf = 0$; an equation $\omega(x, y) = 0$ is invariant if $U\omega = 0$, by virtue of $\omega = 0$; an invariant curve is either 1° a path curve,† *i. e.*, the locus of all the positions which a point takes when subjected to all the transformations of the group, or 2° a curve all of whose points remain absolutely at rest by every transformation of the group. The first are given by equating an arbitrary invariant function to a constant; the second by the common solutions of the equations $\xi = 0$, $\eta = 0$. The first part concludes with a brief study of the projective, conformal and area-preserving point transformations of the plane relative to the points and properties of the preceding sections just enumerated.

* The ∞^1 transformations $x_1 = x + 1$, $y_1 = y + t$ do not form a group; $\tan \pi x$ is an invariant function but not every function of $\tan \pi x$; the family of curves $\tan \pi x = k$ is invariant. The ∞^1 transformations $x_1 = xt$, $y_1 = y + t - 1$ have no invariant function. The latter family contains the identical transformation and an infinitesimal transformation. There is a system of ∞^2 straight lines invariant by the family.

† Bahncurve; trajectoire.

II. In order to successfully apply the theory of groups to differential equations admitting of known infinitesimal transformations there remains still a question to be met—When does a family of ∞^1 curves admit of the transformations of a group? Lie gives the answer in the following theorem: A family of curves, $\omega(x, y) = constant$, admits of the group Uf in the case when $U\omega = \Omega(\omega)$ and in no other. If each curve of the family is invariant, $\Omega = 0$; if the curves of the family are interchanged among themselves, $\Omega \neq 0$; and in the latter case Ω may be taken equal to unity. The preceding theorem combined with the fact that the integral curves, $\omega(x, y) = constant$, of an ordinary differential equation $X(x, y)dy = Y(x, y)dx$, may be defined by the solution ω of the associated homogeneous linear partial differential equation

$$Af \equiv X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial x} = 0,$$

leads to the following theorem first published by Lie in 1874 and establishing the fundamental connection between the integrating factor of Euler and the infinitesimal transformation of Lie: If the differential equation $Xdy - Ydx = 0$ admits of the known infinitesimal transformation $Uf = \xi p + \eta q$, where $X\eta - Y\xi \neq 0$, then

$$\frac{1}{X\eta - Y\xi}$$

is a Eulerian multiplier of the equation and the equation of the integral curves is

$$\int \frac{Xdy - Ydx}{X\eta - Y\xi} = \text{constant}.$$

If $X\eta - Y\xi = 0$, each integral curve is invariant by itself and the transformation Uf is of no avail; Uf in this case is said to be *trivial* with regard to the given equation.

For the practical application of this theorem there is necessary a criterion that a given differential equation admit of an infinitesimal transformation Uf. Lie finds this criterion to lie in the demand that the commutator $(UA)f$ shall be identically equal to $\lambda(x, y) \cdot Af$.

Conversely, if we are given a multiplier M of a differential equation, we shall have an infinitesimal transformation of which the equation admits, by determining ξ and η from the condition $\frac{1}{X\eta - Y\xi} = M$, an indeterminate equation which

shows that every differential equation of the first order in two variables admits of an infinite number of infinitesimal transformations. A similar theorem does not hold for differential equations of a higher order.

If Uf is reduced to its canonical form the equation admitting of it assumes the immediately integrable form $dy' - F(x')\,dx' = 0$. If the equation admits of $U_1 f$ and $U_2 f$ then $(X\eta_1 - Y\xi_1)/(X\eta_2 - Y\xi_2)$ is either an integral of the equation or a constant; finally if the equation admits of the non-trivial transformation Uf it admits of $\Omega(\omega)\,Uf + \varphi(x,y)\,Af$.

On the other hand if we start from the finite equations of the group Uf, we can determine by differentiation and elimination, all ordinary differential equations of the first order in x and y which admit of the group generated by Uf. In this way Lie finds all the known cases of integrability. For example for the homogeneous equation Uf is $xp + yq$ and for the general linear equation Uf is $e^{\int \phi(x)\,dx} \frac{\partial f}{\partial y}$; the finite equations of these groups are $x_1 = ax$, $y_1 = ay$; $x_1 = x$, $y_1 = y + ae^{\int \phi(x)\,dx}$, respectively.

The second part concludes with a chapter on geometrical applications of the theory of integration developed in this part. It includes, among other things, Lie's geometrical interpretation of Euler's multiplier and a number of the theorems relative to curves on surfaces that appeared in Lie's earlier geometrical work.

III. The first chapters of the third part repeat the theorems of the first part for groups in three variables. It has been remarked that the applications were made in the preceding articles to the explicit form $Xdy - Ydx = 0$; by introducing the notion *extended* group Lie makes possible the direct application of these theorems to the general form $\Omega(x,y,y') = 0$. The transformation in the three variables x, y, y'

$$x_1 = \varphi(x,y),\ y_1 = \psi(x,y),\ y_1' = \frac{dy_1}{dx_1} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} = \chi(x,y,y'),$$

Lie defines as the first extension of the point transformation $x_1 = \varphi$, $y_1 = \psi$; it is also called the once extended point transformation. If the transformation belongs to a G_1 the extended transformation belongs to a G_1 which is the extended group of the first G_1. To the infinitesimal transformation $Uf = \xi p + \eta q$ corresponds the once extended infinitesimal transformation

$$U'f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + (\eta_x + y'(\eta_y - \xi_x) - y'^2 \xi_y) \frac{\partial f}{\partial y'}$$

$$\equiv \xi(x,y) \frac{\partial f}{\partial y} + \eta(x,y) \frac{\partial f}{\partial y} + \eta'(x,y,y') \frac{\partial f}{\partial y'} = Uf + \eta' \frac{\partial f}{\partial y'}.$$

A differential equation $\Omega(x, y, y') = 0$ may now be regarded as an algebraic equation in the three variables x, y, y' and the group $U'f$ as a G_1 in the same three variables and we have the criterion: The differential equation $\Omega(x, y, y') = 0$ admits of the group $U'f$, and, therefore, Uf, in the case when $U'\Omega = 0$, either identically, or by virtue of $\Omega = 0$, and in no other case. Conversely all differential equations $\Omega = 0$ which admit of a given group Uf are found by solving the problem of finding all surfaces in x, y, y' which admit of $U'f$. The latter problem is solved in the same way as the problem already handled in the plane. Any two independent solutions u and v of the linear partial differential equation $U'f = 0$ are two invariants and every invariant of the group is a function of these two. The ∞^2 pathcurves are given by the equations $u = const.$ and $v = const.$ There are two kinds of invariant surfaces: 1° those generated by ∞^1 path curves and represented by a single arbitrary equation in u and v; 2° those made up of invariant points. Lie calls an invariant of $U'f$ a differential invariant of the first order of the original group Uf; hence his theorem that every differential equation $\Omega(x, y, y') = 0$ which admits of Uf is found by equating a differential invariant to zero.

IV. Extending the preceding notions to the case of n variables, the necessary and sufficient condition that

$$Af = \sum_1^n a_i \frac{\partial f}{\partial x_i} = 0 \text{ admit of } Uf = \sum_1^n \xi_i \frac{\partial f}{\partial x_i} \text{ is that}$$

$$(UA) \equiv \lambda(x_1, \cdots, x_n) Af.$$

By means of the expression

$$((U_iU_j)A) + ((U_jA)_iU) + ((AU_i)U_j) \equiv 0,$$

which he calls the identity of Jacobi, Lie finds that if $Af = 0$ admits of U_if and U_jf it also admits of $(U_iU_j)f$. If $Af = 0$ admits of $n - 1$ known infinitesimal transformations

$$U_jf = \sum_1^n \xi_{jk} \frac{\partial f}{\partial x_k},$$

which are linearly independent of each other and Af, then a multiplier of Af is given by

$$\frac{1}{M} = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n-1,1} & \xi_{n-1,2} & \cdots & \xi_{n-1,n} \end{vmatrix}.$$

Every differential equation of an order higher than the first does not necessarily admit of an infinitesimal transformation. The most general differential equation of the second order which admits of the infinitesimal transformation or G_1, Uf, is found by equating to zero the most general invariant of the twice extended group

$$U''f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} + \eta'' \frac{\partial f}{\partial y''},$$

where

$$\eta' = \frac{d\eta}{dx} - y' \frac{d\xi}{dx}, \quad \eta'' = \frac{d\eta'}{dx} - y'' \frac{d\xi}{dx}.$$

The most general invariant of $U''f$, which by definition is also a differential invariant of the second order of Uf, is an arbitrary function of u, v, and $w = dv : du$, where u is an invariant of Uf, v a first order differential invariant of Uf to be found by a quadrature if u is known. In a similar manner all differential equations of a higher order which admit of Uf are found by differentiation and quadrature if a zero order differential invariant of Uf is known. The criterion that $\Omega(x, y, y', y'') = 0$ admit of Uf is that $U''\Omega = 0$, either identically, or by virtue of $\Omega = 0$. If $\Omega = 0$ admits of a known Uf its integration demands the integration of two differential equations of the first order and two quadratures. If $\Omega(x, y, y', y'') = 0$ be taken in the solved form $y'' = \omega(x, y, y')$, then the latter admits of Uf, if the equivalent linear partial differential equation

$$Af = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + \omega(x, y, y') \frac{\partial f}{\partial y'} = 0,$$

admits of $U'f$. In the latter event $Af = 0$ and $Uf = 0$ form a so-called complete system, that is they have a solution in common, which according to a theorem of du Bois-Reymond may be found by the integration of an ordinary differential equation of the first order in x, y, and a second solution is given by a quadrature.

A study of differential equations of the second order that admit of several infinitesimal transformations leads to the notion *group of infinitesimal transformations*. r independent infinitesimal transformations $U_1 f, \cdots, U_r f$ form an r parameter group when the commutators $(U_i U_k) = \sum_1^r c_{ik_s} U_s f$; the constants c_{ik_s} determine the structure of the group. If $\rho < r$ of the r infinitesimal transformations $U_1 f, \cdots, U_\rho f$ form a group, the latter is called a ρ parameter subgroup, which, in particular, is an invariant subgroup if the commutators of $U_1 f, \cdots, U_\rho f$ with all the $U_1 f, \cdots, U_r f$ belong to the subgroup. The commutators $(U_i U_k)$ above form an invariant subgroup, the so-called first derived group of the original. Every infinitesimal transformation of the group belongs to at least one two-parameter subgroup which can always be found by algebraic process.

The aggregate of all the infinitesimal transformations of which a differential equation of the second order can admit forms at most an eight-parameter group; the equivalent linear partial differential equation in (x, y, y') admits of the group of the extended infinitesimal transformations; the latter group is determined by the theorem that the extension of a commutator is identical with the commutator of the extended transformations. Hence we see from the preceding that if an equation of the second order admits of two or more independent infinitesimal transformations it admits of at least one two-parameter group. This shows the necessity of reducing the two-parameter groups of the plane to canonical forms. The integration of the equation in these circumstances Lie effects in the two following ways:

1°. By reducing the two-parameter groups of the xy-plane to their four canonical forms: $p, q;\ q, xq;\ q, xp+yq;\ q, yq$; the first three reductions demand at most two quadratures, the last the integration of an ordinary equation of first order; and introducing the reducing or canonical variables in the given equation of the second order it assumes respectively one of the following forms: $y'' - \varphi(y') = 0$; $y'' - \varphi(x) = 0$; $xy'' - \varphi(y') = 0$; $y' - y'\varphi(x) = 0$; whose integration demand at most two quadratures.

2°. By using the equivalent partial differential equation. The integration of a linear partial differential equation $Af = 0$ in the three variables x, y, z that admits of two different infinitesimal transformations involves either two quadratures or the integration of a differential equation of the first order in x, y. From this it follows that the inte-

gration of an ordinary differential equation of the second order admitting of a two-parameter group can be effected by two independent or dependent quadratures at most.

V. The fifth part is chiefly concerned with differential equations admitting of three-parameter groups. By the aid of the notions subgroup, derived group and invariant subgroup referred to in the review of the fourth part, all three-parameter groups of the plane are classified. If the three infinitesimal transformations are U_1f, U_2f, U_3f, the commutators $(U_1U_2)f$, $(U_1U_3)f$, $(U_2U_3)f$ form the first derived group of the original one. There are six different types of structure of three-parameter groups in the plane, one, two, two, one, respectively, according as this first derived group is three, two, one, or no parameter. Further division according to the nature of the path curves gives, all told, thirteen different types of three-parameter groups. These results are interpreted geometrically by representing every infinitesimal transformation of the family of transformations $c_1U_1f + c_2U_2f + c_3U_3f$ by a point in the plane whose homogeneous coordinates are c_1, c_2, c_3. The commutator then expresses the relation between pole and polar with regard to a fixed conic.

The problem of reducing a given three-parameter group to its type in the above scheme of thirteen types falls into two problems: 1° to determine the type; 2° to determine the canonical variables. In Lie's terminology the first problem is to *norm* the given group; the norming of a group in this case is the forming of the first derived group and observing to what type of structure it belongs. The determination of the canonical variables involves only algebraic operations, or, at most, quadratures, except in the case of two unfavorable types whose reduction demands the integration of a differential equation of the first order; but these unfavorable cases do not appear in the integration theory of equations of the second order since there are no such equations admitting of the groups in question. The problem of integrating a differential equation of the second order admitting of a three-parameter group can be solved if the two following admit of solution: 1° to reduce a three-parameter group to its canonical form; 2° to integrate a differential equation of the second order which admits of one of the eleven canonical forms. By the preceding the solution of the first involves only possible operations; the eight different types of equations of the second order admitting of one or more of the eleven canonical forms are immediately integrable; hence the integration of an equation

of the second order admitting of a three-parameter group is effected by possible operations involving no more than quadratures in the most unfavorable cases. In most cases the introduction of the canonical variables can be avoided and the integration performed by operations purely algebraic, if the integration problem be referred to that of the equivalent linear partial differential equation which admits of necessity of the extended group $U_1'f$, $U_2'f$, $U_3'f$; here again the most unfavorable case exacts no more than a quadrature.

The concluding chapter shows how the application of the methods of the book may be made to differential equations of the third order in two variables having known infinitesimal transformations and to partial differential equations of the first order in four variables admitting of three-parameter groups. If the first derived group of the latter has fewer than three parameters the integration is affected by three quadratures, the first two or last two of which are independent; if the first derived group has three parameters the integration of a Riccati equation is demanded.

A paragraph relative to the meaning and importance of the theories in exposition here for the general theory of differential equations, calling attention among other points to analogies with Galois' theory of algebraic equations, concludes this, the introductory volume of Lie's published works.

EDGAR ODELL LOVETT.

PRINCETON, N. J.,
5 *November*, 1897.

SHORTER NOTICE.

Famous Problems of Elementary Geometry. An authorized translation of F. KLEIN'S *Vorträge über ausgewählte Fragen der Elementargeometrie*, by WOOSTER WOODRUFF BEMAN and DAVID EUGENE SMITH. Boston and London, Ginn and Company, 1897. 12mo, pp. ix+80.

Whatever opinion one may hold privately as to the desirability of translations in general, the appearance of a readable English version of Professor Klein's pamphlet* can excite no feeling other than that of satisfaction. This lucid exposition of the historical and actual significance of the three great problems of Greek geometry is addressed to all interested in the teaching of elementary mathematics,

*Leipzig, B. G. Teubner, 1895.

and ought to be in the hands of all; hence we gladly record the fact that Professors Beman and Smith have given us a version that can be read with pleasure. Here and there the translators have perhaps overstepped the legitimate bounds of freedom in the matter of rearrangement; for example, the alteration made in the order of the proof in Part I., Ch. IV., § 6, seems unnecessary. On the other hand, in Part II., Ch. IV., § 1 (p. 68), they might with advantage have availed themselves of the license that they have rightly taken elsewhere, instead of attempting to translate the statement of the nature of the transcendence of e literally; the apparent anacoluthon in the sentence as it stands makes it almost unintelligible without reference to the original. This however seems to be about the only awkward turn in the version, to which as a whole can be given the high praise that no one would suspect it of being a translation from the German. The pages are of a convenient size, and attractive in appearance; the printing is singularly free from errors, and the few obvious ones in the original have been corrected. The little volume in its present dress deserves to be widely circulated among mathematical students at a comparatively early stage of their work.

CHARLOTTE ANGAS SCOTT.

NOTES.

WE have to record the death of ERNST SCHERING, professor of mathematics and director of the magnetic observatory at the University of Göttingen. He died at Göttingen, on November 2, at the age of sixty-four years.

THE mathematical seminar at Göttingen is concerned for this semester with dynamics, much attention being given to the differential equations. The seminar, which is under the guidance of Professors KLEIN and HILBERT, is in connection with Professor KLEIN'S course on mechanics, mention of which was omitted in our list of Göttingen lectures in the BULLETIN for November (see p. 81).

AT the Collège de France, in Paris, two mathematical courses were announced for the term which began December 6th. Professor C. JORDAN offered a course in theory of differential equations; and Professor J. HADAMARD, formerly of the Faculty of Sciences of Bordeaux, but recently appointed deputy professor of mechanics at the Collège de France, offered a course on the curves which satisfy the differential equations of dynamics.

PROFESSOR HERMITE has ceased to lecture at the Sorbonne, and has been given the title of honorary professor.

AT the anniversary meeting of the Royal Society of London, held November 30, one of the Royal medals was awarded to Professor A. R. FORSYTH.

THE Royal Academy of Sciences, Letters and Arts, of Belgium offers six prizes of six hundred francs each to be awarded during 1898. The subject assigned for one of these prizes is "To make an important contribution to the study of the correspondences which can be established between two spaces." The Academy explains that it will accept, for example, a study of the relations between two systems of four homogeneous variables in the sense understood in the researches of Clebsch (See his "Lectures on Geometry," Chapter 7), or, equally well, a geometrical and analytical study of the equation

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{14}x_1x_4 + 2a_{23}x_2x_3 + 2a_{24}x_2x_4 + 2a_{34}x_3x_4 = 0$$

in which the coefficients are functions of the second degree in y_1, y_2, y_3, y_4. Memoirs should be written in either the French or the Dutch language, and should be sent to the secretary of the Academy, Palais des Académies, Brussels, before August 1, 1898.

AMONG the scientific publications announced as in press by G. CARRÉ ET C. NAUD we note the following: "Cours d' Analyse, professé à l'École centrale," by Professor APPELL; "Potentiel newtonien. Leçons professées à la Sorbonne," by Professor POINCARÉ; a new edition of "Electricité et Optique," by Professor POINCARÉ; "Cours de Thermodynamique, professé à la Sorbonne," by Professor LIPPMANN; "Unités électriques et leur mesure en valeur absolue, professé à la Sorbonne," by Professor LIPPMANN. Among the recent publications of the same firm are: A translation of Greenhill's Elliptic Functions by Professor J. GRIESS; "Introduction à l'étude des théories de la mécanique," by HENRI BOUASSE; "Leçons sur l'attraction et la fonction potentielle, professées à la Sorbonne," by Professor APPELL; "Traité élémentaire de cristallographie géométrique," by G. LION; "Leçons de cinématique, professées à la Sorbonne," by P. PUISEUX; "Astronomie sphérique. Notes sur le cours professé à la Sorbonne," by Professor OSSIAN-BONNET; "Exposition de la méthode d'interpolation de Cauchy," by E. CARVALLO; "Traité

d'astronomie théorique, contenant l'exposition du calcul des perturbations planétaires et lunaires et son application à l' explication et à la formation des tables astronomiques," by ABEL SOUCHON.

THE *Revue générale des sciences pures et appliquées* for November 15 contains a full report of the address of Professor POINCARÉ, "The relations between pure analysis and mathematical physics," prepared for the international mathematical congress at Zürich. Among the articles of interest to mathematicians which have appeared in this journal during 1897, are one by JULES TANNERY on mathematical infinity, one by Professor EMILE PICARD on the work of Galois, and a report of the Zürich congress by EMILE BOREL. We observe also two other articles by Professor POINCARÉ, one on the ideas of Hertz in mechanics, the other an annual report on the progress of physics.

PROFESSOR HENRY S. PRITCHETT, formerly head of the department of astronomy at Washington University, St. Louis, and recently appointed Superintendent of the U. S. Coast and Geodetic Survey, assumed the duties of his new office on December 1.

PROFESSOR E. S. HOLDEN, director of the Lick observatory, has resigned. Professor J. M. SCHAEBERLE has been appointed acting director.

THE REV. GEORGE M. SEARLE has resigned as director of the astronomical observatory of the Catholic University of America, Washington, and Mr. Alfred Doolittle has been appointed to succeed him.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BEMAN (W. W.). A chapter in the history of mathematics. An address by vice-president W. W. Beman, chairman of section A, before the section of mathematics and astronomy, American Association for the Advancement of Science, Detroit meeting, 1897. (*Proceedings of the American Association for the Advancement of Science*, vol. 46, 1897.) Salem, Mass., 1897. 8vo. 20 pp.

ENGEL (F.). See STÄCKEL (P.).

JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann, herausgegeben von E. Lampe. Vol. 26: Jahrgang 1895. (In 3 Heften.) Heft 2. Berlin, Reimer, 1897. 8vo. Pp. 529–784. Mk. 6.40

LAMPE (E.). See JAHRBUCH.

LAUGEL (L.). See STÄCKEL (P.).

LAURENT (H.). See STURM (C.).

PROUHET (E.). See STURM (C.).

ROBERTO (G.). Iperdimensioni. Potenza, Garramone e Marchesiello, 1897. 16mo. 12 pp.

SAINT-GERMAIN (A. DE). See STURM (C.).

SPARRE (DE). Sur la réduction aux fonctions elliptiques de certaines intégrales. Paris, Gauthier-Villars, 1897. 8vo. Fr. 0.75

STÄCKEL (P.) et ENGEL (F.). Gauss, les deux Bolyai et la géométrie non euclidienne. Traduit par L. Laugel. Paris, Gauthier-Villars, 1897. 8vo. 4 and 23 pp.

STUDNICKA (F. J.). Beitrag zur Theorie der Potenz- und Combinations-Determinanten. Prag, 1897. 8vo. 20 pp. Mk. 1.50

STURM (C.). Cours d'analyse de l'Ecole polytechnique, revu et corrigé par E. Prouhet, et augmenté de la théorie élémentaire des fonctions elliptiques par H. Laurent. 11e édition, revue et mise au courant du nouveau programme de la licence par A. de Saint-Germain. 2 vols. Paris, Gauthier-Villars, 1897. 8vo. Fr. 15.00

VILLIÉ (E.). Compositions d'analyse, cinématique, mécanique et astronomie, données depuis 1869 à la Sorbonne pour la licence ès sciences mathématiques, suivies d'exercices sur les variables imaginaires. Enoncés et solutions. Partie III : Compositions données depuis 1889. Paris, Gauthier-Villars, 1898. 8vo. Fr. 8.00

II. ELEMENTARY MATHEMATICS.

ANGEL (H.). Practical plane geometry and projection : for science classes, schools and colleges ; adapted to higher stages in Science and art department. Syllabus. Vol. I : Text. 12th thousand. London, Collins, 1897. 8vo. 368 pp. Cloth. 4s.

BERTRAND (G.). Trattato d'aritmetica. Prima traduzione italiana, con note ed aggiunte di G. Novi. 39a ristampa rivista e corretta. Firenze, Le Monnier, 1898. 16mo. 4 and 452 pp. Fr. 3.25

BETTI (E.). See EUCLIDE.

BRIOSCHI (F.). See EUCLIDE.

BÜRKLEN (O.). Lehrbuch der ebenen Trigonometrie mit Beispielen und 280 Uebungsaufgaben für höhere Lehranstalten und zum Selbstunterricht. Heilbronn, Schröder, 1897. 8vo. 7 and 122 pp. Cloth. Mk. 1.50

BUSSLER (F.). Die Elemente der Mathematik, für höhere Lehranstalten bearbeitet. 2te Auflage. Teil 1 : Pensum für die Mittelklassen (Quarta bis Untersecunda). 4 and 151 pp. Teil 2 : Pensum für die Oberklassen (Obersecunda und Prima). 4 and 234 pp. Dresden, Ehlermann, 1897. 8vo. Boards. Mk. 4.10

BUSSLER (F.) Mathematisches Uebungsbuch. Teil 1 : Für den Gebrauch in den mittleren Klassen höherer Lehranstalten (Untertertia bis Untersecunda) zusammengestellt. 2te Auflage. Dresden, Ehlermann, 1897. 8vo. 4 and 88 pp. Boards. Mk. 1.00

COR (N.) et RIEMANN (J.). Traité d'algèbre élémentaire, à l'usage des élèves de mathématiques élémentaires, des aspirants au baccalauréat de l'enseignement classique (2e série) et au baccalauréat de l'enseignement moderne (2e et 3e séries), et des candidats aux écoles du gouvernement. Paris, Nony, 1898. 8vo. 466 pp.

EIBE. SEE EUCLID.

EUCLID. Elementer I–II, ed. Eibe. Köbenhavn, Gyldendal, 1897.

EUCLIDE. Gli elementi, con note, aggiunte ed esercizî ad uso de'ginnasî e de'licei, per cura di E. Betti e F. Brioschi. 32a ristampa. Firenze, Le Monnier, 1898. 16mo. 8 and 60 pp. Fr. 0.75

EUCLIDE. Gli elementi, nuovamente tradotti, con note, aggiunte ed esercizî, ad uso dei ginnasî e dei licei, per cura di A. Socci e G. Tolomei. Lib. VII–X. Firenze, Le Monnier, 1897. 8vo. Pp. 481–665. Fr. 1.50

GIRNDT (M.). Raumlehre, für Baugewerkschulen und verwandte gewerbliche Lehranstalten. Teil 1: Die Lehre von den ebenen Figuren. Mit 227 der Baupraxis entlehnten Aufgaben. Leipzig, Teubner, 1897. 8vo. 8 and 99 pp. Boards. Mk. 2.40

GRASSI (F.). See SERRET (G. A.).

HABERL (J.). Lehrbuch der allgemeinen Arithmetik und Algebra. 7te Auflage. Wien, 1897. 8vo. 6 and 407 pp. Cloth. Mk. 4.00

HOLL (W.). Lehrbuch der Geometrie. Die Lehre von den geometrischen Raumgrössen in geeigneter Verbindung mit Zeichnen und Rechnen. 3te Auflage, herausgegeben von K. Holl. Stuttgart, Kohlhammer, 1897. 8vo. 9 and 131 pp., 76 plates. Boards. Mk. 1.80

KLEYER (A.). Vollständig gelöste Aufgabensammlung aus allen Zweigen der Rechenkunst, der niederen und höheren Mathematik, der Physik, etc. Stuttgart, Maier, 1897. 8vo.

KLEYER (A.) und SACHS (J.). Lehrbuch der ebenen Elementargeometrie (Planimetrie). Nach System Kleyer. Teil VIII: Die Anwendung der Aehnlichkeit auf die Lehre vom Kreis, von J. Sachs. Stuttgart, Maier, 1897. 8vo. 8 and 216 pp., 1 table. Mk. 5.00

NOVI (G.). See BERTRAND (G.).

PIFFL. Aufgabensammlung aus der Algebra mit Berücksichtigung kulturhistorischer, geographischer und naturwissenschaftlicher Daten etc. 2te Auflage. Serajewo, Piffl, 1897.

RIEMANN (J.). See COR (N.).

SACHS (J.). See KLEYER (A.).

SERRETT (G. A.). Trattato di trigonometria piana e sferica, tradotto in italiano sulla settima edizione francese, col consenso del autore, da F. Grassi. 4ta edizione, con note ed aggiunte del traduttore, 1000 esercizî colle risposte e un formulario di matematica e fisica. Firenze, Bocca, 1897. 8vo. 312 pp. Fr. 2.50

SERRETT (G. A.) Trattato di trigonometria. Prima traduzione italiana di A. Ferrucci. 3a edizione, con modificazioni, aggiunte e numerosi esercizî di G. Tolomei. Firenze, Le Monnier, 1897. 8vo. 311 pp. Fr. 2.50

SOCCI (A.). See EUCLIDE.

TOLOMEI (G.). Tavole di logaritmi con cinque decimali. Firenze, Le Monnier, 1897. 16mo. 204 pp. Fr. 1.50

TOLOMEI (G.). See EUCLIDE and SERRET (G. A.).

VERONESE (G.). Appendice ag.i elementi di geometria. Verona e Padova, Drucker, 1898. 16mo. 8 and 89 pp. Fr. 1.50

WEEKS (W.). Exercises in Euclid, graduated and systematised. London, Macmillan, 1897. 18mo. 1s. 9d.

III. APPLIED MATHEMATICS.

BERTHELOT (M.). Thermochimie. Données et lois métriques. Vol. I: Les lois numériques. 17 and 737 pp. Vol. II: Les données expérimentales. 838 pp. Paris, Gauthier-Villars, 1897. 8vo. Fr. 50.00

BLASIUS (C. and E.). See BRAVAIS (A.).

BRAVAIS (A.). Abhandlung über die Systeme von regelmässig auf einer Ebene oder im Raume verteilten Punkten. (1848.) Uebersetzt und herausgegeben von C. und E. Blasius. (Ostwald's Klassiker der exakten Wissenschaften, No. 90.) Leipzig, Engelmann, 1897. 12m. 142 pp., 2 plates. Mk. 2.00

COHN (B.). Ueber die Gauss'sche Methode, aus den Beobachtungen dreier gleichen Sternhöhen die Höhe, Zeit und Polhöhe zu finden, und praktische Hilfsmittel zu ihrer Anwendung. [Diss.] Strassburg, Singer, 1897. 4to. 70 pp., 4 plates. Mk. 4.00

DELLINGSHAUSEN (N.). Grundzüge der kinetischen Naturlehre. Heidelberg, Winter, 1897. 8vo. 8 and 520 pp. Mk. 10.00

DIETSCH (C.). Leitfaden der darstellenden Geometrie; mit sehr vielen Aufgaben. 3te Auflage. Leipzig, Deichert, 1897. 8vo. 4 and 152 pp. Mk. 2.50

FECHNER (G. T.). Kollektivmasslehre. Im Auftrage der königl. sächsischen Gesellschaft der Wissenschaften herausgegeben von G. F. Lipps. Leipzig, Engelmann, 1897. 8vo. 10 and 483 pp. Mk. 14.00

HESS (E.). See HESSEL (J. F. C.).

HESSEL (J. F. C.). Krystallometrie oder Krystallonomie und Krystallographie, auf eigentümliche Weise und mit Zugrundelegung neuer allgemeiner Lehren der reinen Gestaltenkunde, sowie mit vollständiger Berücksichtigung der wichtigsten Arbeiten und Methoden anderer Krystallographen. (1830.) Besonders abgedruckt aus Gehler's physikalischem Wörterbuche. Herausgegeben von E. Hess. (Ostwald's Kassiker der exakten Wissenschaften, Nos. 88, 89.) Leipzig, Engelmann, 1897. 12mo. 192 pp., 8 plates; 165 pp., 3 plates. Mk. 5.80

HUBER (W.). Katechismus der Mechanik. 6te Auflage, neu bearbeitet von W. Lange. Leipzig, Weber, 1897. 8vo. Cloth. Mk. 3.50

KLEIBER (M.). Das projektive Zeichnen nebst den für das Zeichnen wichtigsten Aufgaben aus der ebenen Geometrie. Im Auftrage der königl. Kunstgewerbeschule zu München herausgegeben. 50 zum Teil farbige Vorlageblätter mit begleitendem Text. Für gewerbliche Fortbildungsschulen, Realschulen, Kunstgewerbe-, Baugewerk-, und Industrie-Schulen, sowie zum Selbstunterricht. 2te Auflage. Stuttgart, Effenberger, 1897. Folio, 8 and 92 pp., 50 plates. Mk. 12.00

KRÄMER (J.). See MEISSNER (G.).

KRAUSS (F.). Graphische Kalorimetrie der Dampfmaschinen. Berlin, Springer, 1897. 8vo. 6 and 67 pp., 1 plate. Mk. 2.00

LA GOURNERIE (DE). Traité de perspective linéaire. 3e édition, revue et corrigée, avec une introduction de E. Lebon. Paris, Gauthier-Villars, 1898. 1 volume in 4to, with an atlas, in folio, of 40 plates. Fr. 25.00

LANGE (W.). See HUBER (W.).

———. Katechismus der Statistik, mit gesonderter Berücksichtigung der zeichnerischen und rechnerischen Methoden. Leipzig, Weber, 1897. 8vo. Cloth. Mk. 4.00

LEBON (E.). See LA GOURNERIE (DE).

LIPPS (G. F.). See FECHNER (G. T.).

LIPPS (T.). Raumaesthetik und geometrisch-optische Täuschungen. Leipzig, 1897. 8vo. 8 and 424 pp., 1 plate. Mk. 12.00

MEISSNER (G.). Die Kraftübertragung auf weite Entfernungen und die Konstruktion der Triebwerke und Regulatoren. 2te Auflage, von J. Krämer. Vol. I. Jena, Costenoble, 1897. 8vo. 387 pp., 30 plates. Mk. 18.00

PARETO (V.). La courbe de la répartition de la richesse. Lausanne, 1897. 4to. 18 pp. Mk. 1.50

PECHAN (J.). Berechnung der Leistung und des Dampfverbrauchs der Zweicylinder-Dampfmaschinen zweistufiger Expansion. Wien, Deuticke, 1897. 8vo. 15 and 289 pp., 48 tables. Mk. 8.00

SCHLOTKE (J.). Lehrbuch der darstellenden Geometrie. (In 4 Teilen.) Teil I: Specielle darstellende Geometrie. 3te, verbesserte Auflage. Dresden, Kühtmann, 1897. 8vo. 4 and 154 pp. Mk. 3.60

SCHMIDT-ULM (G.). Die Wirkungsweise, Berechnung und Konstruktion der Gleichstrom-Dynamomaschinen und Motoren. Leipzig, Leiner, 1897. 8vo. 8 and 272 pp., 34 plates. Mk. 8.50

SCHULTZ (E.). Vierstellige mathematische Tabellen (Ausgabe A) für gewerbliche Lehranstalten. 2te Auflage. 8vo. 5 and 80 pp. Nebst einer Anleitung zum Gebrauche der mathematischen Tabellen in den technischen Kalendern, an 25 Beispielen aus der Praxis erläutert. 2te Auflage. 16mo. 31 pp. Essen, Baedeker, 1897. Cloth. Mk. 1.20

WÜLLNER (A.). Lehrbuch der Experimentalphysik. 5te Auflage. Vol. III: Die Lehre vom Magnetismus und von der Elektricität, mit einer Einleitung: Grundzüge der Lehre vom Potential. Leipzig, Teubner, 1897. 8vo. 15 and 1414 pp. Mk. 18.00